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## INTEGRAL DISTANCES

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In the present note we are going to prove the following result:

*For any  $n$  we can find  $n$  points in the plane not all on a line such that their distances are all integral, but it is impossible to find infinitely many points with integral distances (not all on a line).<sup>1</sup>*

PROOF. Consider the circle of diameter 1,  $x^2 + y^2 = 1/4$ . Let  $p_1, p_2, \dots$  be the sequence of primes of the form  $4k + 1$ . It is well known that

$$p_i^2 = a_i^2 + b_i^2, \quad a_i \neq 0, \quad b_i \neq 0,$$

is solvable. Consider the point (on the circle  $x^2 + y^2 = 1/4$ ) whose distance from  $(-1/2, 0)$  is  $b_i/p_i$ . Denote this point by  $(x_i, y_i)$ . Consider the sequence of points  $(-1/2, 0), (1/2, 0), (x_i, y_i), i = 1, 2, \dots$ . We shall show that any two distances are rational. Suppose this has been shown for all  $i < j$ . We then prove that the distance from  $(x_j, y_j)$  to  $(x_i, y_i)$  is rational. Consider the 4 concyclic points  $(-1/2, 0), (1/2, 0), (x_i, y_i), (x_j, y_j)$ ; 5 distances are clearly rational, and then by Ptolemy's theorem the distance from  $(x_i, y_i)$  to  $(x_j, y_j)$  is also rational. This completes the proof. Thus of course by enlarging the radius of the circle we can obtain  $n$  points with integral distances.

It is very likely that these points are dense in the circle  $x^2 + y^2 = 1/4$ , but this we can not prove. It is easy to obtain a set which is dense on  $x^2 + y^2 = 1/4$  such that all the distances are rational. Consider the

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<sup>1</sup> Anning gave 24 points on a circle with integral distances. Amer. Math. Monthly vol. 22 (1915) p. 321. Recently several authors considered this question in the Mathematical Gazette.

point  $x_1$  whose distance from  $(-1/2, 0)$  is  $3/5$ ; the distance from  $(0, 1/2)$  is of course  $4/5$ . Denote  $(-1/2, 0)$  by  $P_1$ ,  $(1/2, 0)$  by  $P_2$ , and let  $\alpha$  be the angle  $P_2P_1X_1$ .  $\alpha$  is known to be an irrational multiple of  $\pi$ . Let  $x_i$  be the point for which the angle  $P_1P_2X_i$  equals  $i\alpha$ ; the points  $X_i$  are known to be dense on the circle  $X^2 + y^2 = 1/2$ , and all distances between  $x_i$  and  $x_j$  are rational because if  $\sin \alpha$  and  $\cos \alpha$  are rational, clearly  $\sin i\alpha$  and  $\cos i\alpha$  are also rational.

To give another configuration of  $n$  points with integral distances, let  $m^2$  be an odd number with  $d$  divisors, and put

$$m^2 = x_i^2 - y_i^2$$

This equation has clearly  $d$  solutions. Consider now the points

$$(m, 0), \quad (0, y_i) \quad i = 1, 2, \dots$$

It is immediate that all the distances are integral.

These configurations are all of very special nature. Several years ago Ulam asked whether it is possible to find a dense set in the plane such that all the distances are rational. We do not know the answer.

Now we prove that we cannot have infinitely many points  $P_1, P_2, \dots$  in the plane not all on a line with all the distances  $P_iP_j$  being integral.

First we show that no line  $L$  can contain infinitely many points  $Q_1, Q_2, \dots$ . Let  $P$  be a point not on  $L$ ,  $Q_i$  and  $Q_j$  two points very far away from  $P$  and very far from each other. Put  $d(PQ_i) = a$ ,  $d(Q_iQ_j) = b$ ,  $d(PQ_j) = c$ . ( $d(A, B)$  denotes the distance from  $A$  to  $B$ .)

$$(1) \quad c \leq a + b - 1.$$

Let  $Q_iR$  be perpendicular to  $PQ_i$ . We have

$$a < d(PR) + (d(Q_iR))^2/d(PR), \quad b < d(Q_jR) + (d(Q_iR))^2/d(Q_jR).$$

Thus from (1)

$$(d(Q_iR))^2 \left( \frac{1}{d(PR)} + \frac{1}{d(Q_jR)} \right) > 1$$

which is clearly false for  $a$  and  $b$  sufficiently large. ( $d(Q_iR)$  is clearly less than the distance of  $P$  from  $L$ .) This completes the proof.

There clearly exists a direction  $P_1X$  such that in every angular neighborhood of  $P_1X$  there are infinitely many  $P_i$ .

Let  $P_2$  be a point not on the line  $P_1X$ .

Denote the angle  $XP_1P_2$  by  $\alpha$ ,  $0 < \alpha < \pi$ . Evidently the  $P_i$  cannot form a bounded set. Let  $Q$  be one of the  $P_i$  sufficiently far away from

$P_1$ , where the angle  $QP_1X$  equals  $\epsilon$  ( $\epsilon$  sufficiently small). Denote  $d(P_1, P_2) = a$ ,  $d(P_1, Q) = b$ ,  $d(P_2, Q) = c$ . We evidently have

$$c^2 = a^2 + b^2 - 2ab \cos(\alpha - \epsilon).$$

$a, b, c$  all are integers. From this we shall show that if  $b$  and  $c$  are sufficiently large,  $\epsilon$  sufficiently small, then

$$(2) \quad c = b - a \cos \alpha.$$

Put

$$c = b - a \cos \alpha + \delta, \quad \delta > 0.$$

Then

$$(b - a \cos \alpha + \delta)^2 = b^2 - 2ab \cos \alpha + a^2 \cos^2 \alpha + 2\delta(b - a \cos \alpha) + \delta^2 > a^2 + b^2 - 2ab \cos(\alpha - \epsilon)$$

if  $b$  is sufficiently large and  $\epsilon$  sufficiently small. Similarly we dispose of the case  $\delta < 0$ . Thus (2) is proved.

From (2) we have

$$a^2 + b^2 - 2ab \cos(\alpha - \epsilon) = b^2 - 2ab \cos \alpha + a^2 \cos^2 \alpha$$

or

$$\cos(\alpha - \epsilon) - \cos \alpha = \frac{a^2 \sin^2 \alpha}{2b}.$$

Thus we clearly obtain

$$\epsilon < c_1/b.$$

Thus clearly all the points  $Q_i$  have distance less than  $c_2$  from the line  $P_1X$ . Let  $Q_1, Q_2, Q_3$  be three such points not on a line, where  $d(Q_i Q_j)$  are large. Let  $Q_1 Q_3$  be the largest side of the triangle  $Q_1 Q_2 Q_3$ . Let  $Q_2 R$  be perpendicular to  $Q_1 Q_3$ . We have as before

$$d(Q_1 Q_3) \leq d(Q_1, Q_2) + d(Q_2 Q_3) - 1;$$

also

$$d(Q_1 Q_2) - d(Q_1 R) < \epsilon, \quad d(Q_2 Q_3) - d(Q_3 R) < \epsilon$$

an evident contradiction; this completes the proof.

By a similar argument we can show that we cannot have infinitely many points in  $n$ -dimensional space not all on a line, with all the distances being integral.